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The parafermion Fock space and explicit $\mathfrak{so}(2n+1)$ representations

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Abstract

The defining relations (triple relations) of n pairs of parafermion operators $f_j^{\pm}$ $(j = 1, \ldots, n)$ are known to coincide with a set of defining relations for the Lie algebra $\mathfrak{so}(2n+1)$ in terms of $2n$ generators. With the common Hermiticity conditions, this means that the 'parafermions of order p' correspond to a finite-dimensional unitary irreducible representation $W(p)$ of $\mathfrak{so}(2n+1)$, with highest weight $\left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}\right)$. Although the dimension and the character of $W(p)$ is known by classical formulae, there is no explicit basis of $W(p)$ available in which the parafermion operators have a natural action. In this paper we construct an orthogonal basis for $W(p)$, and present the explicit actions of the parafermion generators on these basis vectors. We use group theoretical techniques, in which the $\mathfrak{u}(n)$ subalgebra of $\mathfrak{so}(2n+1)$ plays a crucial role: a set of Gelfand–Zetlin patterns of $\mathfrak{u}(n)$ will be used to label the basis vectors of $W(p)$, and also in the explicit action (matrix elements) certain $\mathfrak{u}(n)$ Clebsch–Gordan coefficients are essential.

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1. Introduction

It has been known for a long time that Fermi–Dirac and Bose–Einstein statistics do not yield all possible descriptions of particle systems [7, 11]. The classical work of Green [7] is nowadays considered as the basic underlying mathematical formulation of the problem of generalized quantum statistics. In this paper, the notions of Bose oscillators or bosons and Fermi oscillators or fermions are generalized to parabosons and parafermions. Parabosons and parafermions

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are of interest in many applications, in particular in the quantum field theory [5, 21, 23] and generalizations of quantum statistics (parastatistics) [7, 11–13, 16].

The 'creation and annihilation operators' for parabosons or parafermions satisfy certain triple relations, which can be considered as their defining relations. The main object to construct is then the generalization of the boson or fermion Fock space, i.e. the paraboson or parafermion Fock space. Such a Fock space (with unique vacuum vector) is characterized by a certain parameter p, known as the order of parastatistics. One way to construct the Fock space of order p was already discussed by Green, and is referred to as the so-called Green ansatz [7]. The mathematical equivalent is related to finding a proper basis of an irreducible constituent of a p-fold tensor product [11, 16], in fact a superalgebra tensor product for parabosons [16, 26]. The computational difficulties arising here turn out to be very hard, and did not lead to a complete solution of the problem (constructing a proper orthonormal basis of the Fock space, with explicit actions of the paraboson or parafermion operators on this basis).

In a recent paper [17], this problem of giving a complete construction of the Fock space of order p was solved for the paraboson case. The solution relies on an important observation by Ganchev and Palev [6], who showed that the triple relations for n pairs of paraboson operators are the defining relations for the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [14]. As a consequence, the paraboson Fock space of order p is a certain infinite-dimensional unitary irreducible representation (unirrep) of $\mathfrak{osp}(1|2n)$. The construction of this representation was performed in [17], using group theoretical techniques and in particular the branching $\mathfrak{osp}(1|2n) \supset \mathfrak{sp}(2n) \supset \mathfrak{u}(n)$. The result is a complete description of a proper basis (an orthogonal Gelfand–Zetlin-basis) and the explicit action (matrix elements) of the paraboson operators in this basis [17].

In the present paper we tackle the same problem for the case of parafermions. It is already known for a long time that the triple relations for n pairs of parafermion operators are the defining relations for the orthogonal Lie algebra $\mathfrak{so}(2n+1)$ [15, 27]. The parafermion Fock space of order p then corresponds to a unirrep of $\mathfrak{so}(2n+1)$, namely to the finite-dimensional unirrep $W(p)$ with highest weight $\left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}\right)$. Contrary to infinite-dimensional unirreps of the Lie superalgebra $\mathfrak{osp}(1|2n)$, where little was known, a lot is known about finite-dimensional unirreps of Lie algebras and in particular of $\mathfrak{so}(2n+1)$. For example, it is easy to write down the character of $W(p)$ and compute its dimension. On the other hand, a proper basis of $W(p)$, in which the action of the parafermion operators can be given explicitly, is not known. The celebrated Gelfand–Zetlin (GZ) basis for finite-dimensional irreducible representations of the orthogonal Lie groups or Lie algebras, given in [9], is in fact not appropriate here. In the GZ-basis of [9] for $\mathfrak{so}(2n+1)$, the action is given for a set of $2n$ generators, which are however different from the parafermion operators (and also different from the Chevalley generators; in fact, the generators in [9] are not root vectors). The relations between the parafermion operators and the generators used in [9] or the Chevalley generators are highly nonlinear [25], and hence do not lead to a practical and explicit expression of the parafermion matrix elements.

Previous attempts to solve this problem followed essentially Green's ansatz. In terms of the Lie algebra $\mathfrak{so}(2n+1)$, this amounts to constructing $W(p)$ as one of the irreducible components in the p-fold tensor product $W(1)^{\otimes p}$ of $W(1)$, where $W(1)$ is the $\mathfrak{so}(2n+1)$ unirrep with highest weight $\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$, i.e. $W(1)$ is the usual fermion Fock space of dimension 2^n (the spinor representation of $\mathfrak{so}(2n+1)$). In a number of papers, further attention was given to the actual decomposition of $W(1)^{\otimes p}$ into its irreducible components [3, 10, 24]. In [3, 24], the 'vacuum subspaces' of the irreducible components of $W(1)^{\otimes p}$ were considered, and shown to correspond to unirreps of the subalgebra $\mathfrak{u}(n)$. These approaches did not lead to solution for the parafermion matrix elements, however.

In the current paper we present an explicit solution to this problem. Our solution is based upon group theoretical techniques: we use, in particular, the branching $\mathfrak{so}(2n+1) \supset \mathfrak{u}(n)$, an induced representation construction, the known Gelfand–Zetlin basis description for $\mathfrak{u}(n)$ [8], $\mathfrak{u}(n)$ Clebsch–Gordan coefficients, the method of reduced matrix elements and the computational techniques developed in [17]. Our main result is a specific orthogonal basis for $W(p)$ described by means of GZ-patterns, and the explicit action (matrix elements) of the parafermion operators in this basis.

The structure of the paper is as follows. In section 2 we give the defining relations for the parafermion operators, and for the parafermion Fock space $W(p)$. In section 3 we relate the parafermion operators to a set of generators for the Lie algebra $\mathfrak{so}(2n+1)$, and we give a character formula and a dimension formula for $W(p)$. Section 4 is the core of the paper: using an induced module construction, we find a proper basis for $W(p)$ and compute the explicit action of the parafermion creation and annihilation operators on the basis vectors. An example is given in section 5, and we conclude the paper with a summary and some final remarks in section 6.

2. The parafermion Fock space $W(p)$

For a system consisting of n fermions, with creation and annihilation operators $F_i^{\pm}$ $(i = 1, 2, \ldots, n)$, the defining relations are written by means of anticommutators,

$$\left\{F_i^-, F_j^+\right\} = \delta_{ij}, \qquad \{F_i^-, F_j^-\} = \left\{F_i^+, F_j^+\right\} = 0. \tag{2.1}$$

The n-fermion Fock space is defined as a Hilbert space with vacuum vector $|0\rangle$, with

$$\langle 0|0\rangle = 1, \qquad F_i^-|0\rangle = 0, \qquad \left(F_i^{\pm}\right)^{\dagger} = F_i^{\mp} \qquad (i = 1, \ldots, n). \tag{2.2}$$

This Hilbert space is finite-dimensional and irreducible under the action of the algebra spanned by the elements 1, F_i^+, F_i^- $(i = 1, \ldots, n)$, subject to (2.1). A set of orthogonal basis vectors of this space is given by

$$|\theta_1, \ldots, \theta_n\rangle = \left(F_1^+\right)^{\theta_1} \cdots \left(F_n^+\right)^{\theta_n}|0\rangle, \qquad \theta_1, \ldots, \theta_n \in \{0, 1\}. \tag{2.3}$$

This Fock space, denoted by $W(1)$, has dimension 2^n. We shall see that it is a certain unirrep of the Lie algebra $\mathfrak{so}(2n+1)$, with highest weight $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ (the spinor representation).

The purpose of this paper is to study Fock representations of a system of n parafermions. For such a system, with creation and annihilation operators $f_i^{\pm}$ $(i = 1, \ldots, n)$, the defining relations are usually given by

$$\begin{aligned}
&\left[f_j^-, \left[f_k^+, f_l^-\right]\right] = 2\delta_{jk} f_l^-, \\
&\left[f_j^-, \left[f_k^+, f_l^+\right]\right] = 2\delta_{jk} f_l^+ - 2\delta_{jl} f_k^+, \\
&\left[f_j^-, \left[f_k^-, f_l^-\right]\right] = 0
\end{aligned}$$

and their conjugates. They can be written in the following unified form:

$$\left[\left[f_j^{\xi}, f_k^{\eta}\right], f_l^{\epsilon}\right] = \tfrac{1}{2}(\epsilon - \eta)^2 \delta_{kl} f_j^{\xi} - \tfrac{1}{2}(\epsilon - \xi)^2 \delta_{jl} f_k^{\eta}, \tag{2.4}$$

where $j, k, l \in \{1, 2, \ldots, n\}$ and $\eta, \epsilon, \xi \in \{+, -\}$ (to be interpreted as $+1$ and -1 in the algebraic expressions $\epsilon - \xi$ and $\epsilon - \eta$).

The parafermion Fock space $W(p)$ is the Hilbert space with unique vacuum vector $|0\rangle$, defined by means of $(j, k = 1, 2, \ldots, n)$

$$\langle 0|0\rangle = 1, \qquad f_j^-|0\rangle = 0, \qquad \left(f_j^{\pm}\right)^{\dagger} = f_j^{\mp}, \qquad \left[f_j^-, f_k^+\right]|0\rangle = p\delta_{jk}|0\rangle, \tag{2.5}$$

and by irreducibility under the action of the algebra spanned by the elements f_j^+, f_j^- $(j = 1, \ldots, n)$, subject to (2.4). The parameter p is known as the order of the parafermion

system. For $p = 1$ the parafermion Fock space $W(p)$ coincides with the fermion Fock space $W(1)$. In order to understand the structure of $W(p)$ for general (integer) p-values, it will be necessary to make the connection between the algebraic relations (2.4) and the defining relations of the Lie algebra $\mathfrak{so}(2n+1)$.

3. The Lie algebra $\mathfrak{so}(2n+1)$ and its unirrep $W(p)$

The orthogonal Lie algebra $\mathfrak{so}(2n+1)$ consists of matrices of the form

$$\begin{pmatrix} a & b & c \\ d & -a^t & \mathrm{e} \\ -\mathrm{e}^t & -c^t & 0 \end{pmatrix}, \tag{3.1}$$

where a is any $(n \times n)$-matrix, b and d are antisymmetric $(n \times n)$-matrices, and c and e are $(n \times 1)$-matrices (t stands for transpose). Denote by e_{ij} the matrix with zeros everywhere except a 1 on position (i, j) (where $i, j \in \{1, \ldots, 2n+1\}$). Then the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}(2n+1)$ is spanned by the diagonal elements

$$h_j = e_{jj} - e_{n+j,n+j} \qquad (j = 1, \ldots, n). \tag{3.2}$$

In terms of the dual basis ϵ_j of $\mathfrak{h}^*$ the root vectors and corresponding roots of $\mathfrak{so}(2n+1)$ are given by:

$$\begin{aligned}
&e_{jk} - e_{k+n,j+n} \leftrightarrow \epsilon_j - \epsilon_k, && j \neq k = 1, \ldots, n, \\
&e_{j,k+n} - e_{k,j+n} \leftrightarrow \epsilon_j + \epsilon_k, && j < k = 1, \ldots, n, \\
&e_{j+n,k} - e_{k+n,j} \leftrightarrow -\epsilon_j - \epsilon_k, && j < k = 1, \ldots, n, \\
&e_{j,2n+1} - e_{2n+1,j+n} \leftrightarrow \epsilon_j, && j = 1, \ldots, n, \\
&e_{n+j,2n+1} - e_{2n+1,j} \leftrightarrow -\epsilon_j, && j = 1, \ldots, n.
\end{aligned}$$

The positive roots are given by $\Delta_+ = \{\epsilon_j (j = 1, \ldots, n); \epsilon_j - \epsilon_k, \epsilon_j + \epsilon_k (1 \leqslant j < k \leqslant n)\}$. Then it is easy to verify that the following multiples of some of the root vectors

$$f_k^+ = \sqrt{2}(e_{k,2n+1} - e_{2n+1,n+k}), \qquad f_k^- = \sqrt{2}(e_{2n+1,k} - e_{n+k,2n+1}) \qquad (k = 1, \ldots, n) \tag{3.4}$$

satisfy the defining triple relations of n parafermions (2.4). Furthermore, note that all other root vectors of $\mathfrak{so}(2n+1)$ can be obtained by considering all possible commutators $\left[f_j^\xi, f_k^\eta\right]$. This was one of the main results of [15, 27]:

Theorem 1. *As a Lie algebra defined by generators and relations, $\mathfrak{so}(2n+1)$ is generated by the $2n$ elements $f_k^\pm$ subject to the parafermion relations (2.4).*

The parafermion operators f_j^+ are part of the positive root vectors, and the f_j^- are part of the negative root vectors. The generating vector $|0\rangle$ of the parafermion Fock space $W(p)$ satisfies $(j, k = 1, \ldots, n)$

$$f_j^-|0\rangle = 0, \qquad \left[f_j^-, f_k^+\right]|0\rangle = p\delta_{jk}|0\rangle. \tag{3.5}$$

Furthermore, it is easy to verify that

$$\left[f_j^-, f_j^+\right] = -2h_j \qquad (j = 1, \ldots, n), \tag{3.6}$$

from which one deduces $h_j|0\rangle = -\frac{p}{2}|0\rangle$. Hence we have the following:

Corollary 2. *The parafermion Fock space $W(p)$ is the unitary irreducible representation of $\mathfrak{so}(2n+1)$ with lowest weight $\left(-\frac{p}{2}, -\frac{p}{2}, \ldots, -\frac{p}{2}\right)$.*

A lot is known about finite-dimensional representations of the Lie algebra $\mathfrak{so}(2n+1)$ (or of any simple Lie algebra). In particular, dimension and character formulae are available for such representations. In these formulae, one uses the Weyl group $\mathcal{W}$ and the Weyl tool ρ. For the case of $\mathfrak{so}(2n+1)$, $\mathcal{W} = S_n \times \{-1,+1\}^n$, of order $n! \cdot 2^n$ (S_n is the symmetric group). And from (3.3) one can verify that the Weyl tool is

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha = \sum_{j=1}^{n}\frac{2n+1-2j}{2}\epsilon_j = \left(\frac{2n-1}{2},\ldots,\frac{5}{2},\frac{3}{2},\frac{1}{2}\right).$$

Weyl's character formula for an irreducible representation $V(\Lambda)$ with highest weight Λ reads [29]

$$\operatorname{char} V(\Lambda) = \sum_{w\in\mathcal{W}}\varepsilon(w)\,\mathrm{e}^{w(\Lambda+\rho)}\Big/\sum_{w\in\mathcal{W}}\varepsilon(w)\,\mathrm{e}^{w(\rho)}, \tag{3.7}$$

where $\varepsilon(w) = \pm 1$ is the signature of w, and e stands for the formal exponential. Usually, one denotes e^{ϵ_j} by x_j; then the character is a (symmetric) polynomial in the variables $x_1, x_2, \ldots, x_n$.

In the current case, $W(p)$ is an $\mathfrak{so}(2n+1)$ representation with lowest weight $\left(-\frac{p}{2},-\frac{p}{2},\ldots,-\frac{p}{2}\right)$. If p is a positive integer, this is the lowest weight of the finite-dimensional irreducible representation with highest weight $\Lambda = \left(\frac{p}{2},\frac{p}{2},\ldots,\frac{p}{2}\right)$. Note that, with $(\epsilon_1-\epsilon_2,\ldots,\epsilon_{n-1}-\epsilon_n,\epsilon_n)$ as a simple root system, the Dynkin labels of this highest weight are $[0,0,\ldots,0,p]$. Let us compute the character of this representation: splitting the Weyl group into its symmetric group and $\{-1,+1\}^n$, the numerator in (3.7) becomes

$$\sum_{w\in\mathcal{W}}\varepsilon(w)\,\mathrm{e}^{w(\Lambda+\rho)} = \sum_{w\in S_n}\varepsilon(w)w\left(\prod_{j=1}^{n}\left(x_j^{p/2+(2n+1-2j)/2} - x_j^{-p/2-(2n+1-2j)/2}\right)\right),$$

in terms of a determinant, this can be written as

$$\begin{aligned}\sum_{w\in\mathcal{W}}\varepsilon(w)\,\mathrm{e}^{w(\Lambda+\rho)} &= \det_{1\leqslant i,j\leqslant n}\left(x_i^{p/2+(2n+1-2j)/2} - x_i^{-p/2-(2n+1-2j)/2}\right)\\ &= (x_1\cdots x_n)^{-p/2-(2n-1)/2}\det\left(x_i^{p+2n-j} - x_i^{j-1}\right).\end{aligned}$$

Hence, one obtains

$$\operatorname{char} W(p) = (x_1\cdots x_n)^{-p/2}\frac{\det\left(x_i^{p+2n-j} - x_i^{j-1}\right)}{\det\left(x_i^{2n-j} - x_i^{j-1}\right)}. \tag{3.8}$$

The quotient of determinants appearing here has a nice expression in terms of Schur functions $s_\lambda(x)$, see [20, p 84, equation (2′)]. It reads:

$$\operatorname{char} W(p) = (x_1\cdots x_n)^{-p/2}\sum_{\ell(\lambda')\leqslant p}s_\lambda(x_1,\ldots,x_n). \tag{3.9}$$

Herein, the Schur functions $s_\lambda(x)$ are, as usual [20], labeled by a partition $\lambda = (\lambda_1,\lambda_2,\ldots)$, where all λ_i are nonnegative integers and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$. The number of nonzero parts λ_i is the *length* of λ, denoted by $\ell(\lambda)$. As $s_\lambda(x) = s_\lambda(x_1,\ldots,x_n)$ with only n variables x_i, only those λ with $\ell(\lambda) \leqslant n$ appear (since for $\ell(\lambda) > n$ one has $s_\lambda(x_1,\ldots,x_n) = 0$). In (3.9), λ' stands for the *conjugate partition* of λ (a partition whose Young diagram [20] is the transpose of the Young diagram of λ). Otherwise said, the sum in (3.9) is over all partitions λ whose Young diagram fits inside the $(n\times p)$ rectangle. In what follows, expansion (3.9) will be very relevant. For a more general context in which identities such as (3.9) appear, see [22] (the expression (3.9) appears there as [22, theorem 2.3(1)]). The origin of (3.9) goes back to

work of Bracken and Green [2], who do not give (3.9) explicitly, but obtain it implicitly by describing the branching $\mathfrak{so}(2n+1) \supset \mathfrak{u}(n)$ for $W(p)$.

Apart from the character formula, there is also Weyl's dimension formula [29] for an irreducible representation $V(\Lambda)$ with highest weight Λ:

$$\dim V(\Lambda) = \frac{\prod_{\alpha\in\Delta_+}\langle\Lambda+\rho|\alpha\rangle}{\prod_{\alpha\in\Delta_+}\langle\rho|\alpha\rangle}. \tag{3.10}$$

Using (3.3) and $\langle\epsilon_i|\epsilon_j\rangle = \delta_{ij}$, this yields

$$\dim W(p) = \prod_{1\leqslant j\leqslant k\leqslant n} \frac{p+1+2n-j-k}{1+2n-j-k} = \prod_{i=0}^{\lfloor (n-1)/2\rfloor} \frac{\binom{p+2n-2i-1}{2n-4i-1}}{\binom{2n-2i-1}{2n-4i-1}}. \tag{3.11}$$

In the last expression with binomial coefficients, the upper limit for i is the largest integer not exceeding $(n-1)/2$.

The character and dimension of $W(p)$ give already a lot of information, but not yet what we really want, namely an explicit orthogonal basis of $W(p)$ and the action of $f_i^{\pm}$ on these basis vectors. In order to obtain this, we shall follow the induced module construction used in [17].

4. The construction of $W(p)$

From (3.3) it is already clear that the $2n$ elements $f_j^{\pm}$ are a set of generators for $\mathfrak{so}(2n+1)$. In fact, all basis elements of $\mathfrak{so}(2n+1)$ (all root vectors plus a basis of the Cartan subalgebra) are given by

$$f_j^{\pm}(1\leqslant j\leqslant n), \qquad \left[f_j^{\pm}, f_k^{\pm}\right](1\leqslant j<k\leqslant n), \qquad \left[f_j^{+}, f_k^{-}\right] \qquad (1\leqslant j,k\leqslant n). \tag{4.1}$$

Some relevant subalgebras of $\mathfrak{so}(2n+1)$ are easy to describe by means of these parafermion generators $f_j^{\pm}$:

Proposition 3. *A basis for the subalgebra* $\mathfrak{so}(2n)$ *of* $\mathfrak{so}(2n+1)$ *is given by the elements*

$$\left[f_j^{\pm}, f_k^{\pm}\right](1\leqslant j<k\leqslant n), \qquad \left[f_j^{+}, f_k^{-}\right] \qquad (1\leqslant j,k\leqslant n). \tag{4.2}$$

The n^2 *elements*

$$\left[f_j^{+}, f_k^{-}\right] \qquad (1\leqslant j,k\leqslant n) \tag{4.3}$$

are a basis for the $\mathfrak{so}(2n)$ *subalgebra* $\mathfrak{u}(n)$.

In fact, with $\left[f_j^{+}, f_k^{-}\right] = 2E_{jk}$, the triple relations (2.4) imply the relations $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$. In other words, the elements $\left[f_j^{+}, f_k^{-}\right]$ form, up to a factor 2, the standard $\mathfrak{u}(n)$ or $\mathfrak{gl}(n)$ basis elements.

So the parafermion generators $f_j^{\pm}$ highlight the subalgebra chain $\mathfrak{so}(2n+1) \supset \mathfrak{so}(2n) \supset \mathfrak{u}(n)$. We use here the notation $\mathfrak{u}(n)$, algebraically the same as the general linear Lie algebra $\mathfrak{gl}(n)$, but with the conditions $\left(f_j^{\pm}\right)^{\dagger} = f_j^{\mp}$ implying that we are dealing with the 'compact form' $\mathfrak{u}(n)$.

The subalgebra $\mathfrak{u}(n)$ can be extended to a parabolic subalgebra [30] $\mathcal{P}$ of $\mathfrak{so}(2n+1)$:

$$\mathcal{P} = \mathrm{span}\left\{\left[f_j^{+}, f_k^{-}\right](1\leqslant j,k\leqslant n), f_j^{-}(1\leqslant j\leqslant n), [f_j^{-}, f_k^{-}](1\leqslant j<k\leqslant n)\right\}. \tag{4.4}$$

Recall that $\left[f_j^{-}, f_k^{+}\right]|0\rangle = p\delta_{jk}|0\rangle$ with $\left[f_j^{-}, f_j^{+}\right] = -2h_j$. This means that the space spanned by $|0\rangle$ is a trivial one-dimensional $\mathfrak{u}(n)$ module $\mathbb{C}|0\rangle$ of weight $\left(-\frac{p}{2}, \ldots, -\frac{p}{2}\right)$. Since

$f_j^-|0\rangle = 0$, the module $\mathbb{C}|0\rangle$ can be extended to a one-dimensional $\mathcal{P}$ module. Now we can define the Verma module or the induced $\mathfrak{so}(2n+1)$ module $\overline{W}(p)$:

$$\overline{W}(p) = \mathrm{Ind}_{\mathcal{P}}^{\mathfrak{so}(2n+1)}\mathbb{C}|0\rangle. \tag{4.5}$$

This is an $\mathfrak{so}(2n+1)$ representation with lowest weight $\left(-\frac{p}{2},\ldots,-\frac{p}{2}\right)$. By the Poincaré–Birkhoff–Witt theorem [4], it is easy to give a basis for $\overline{W}(p)$:

$$\begin{aligned}&\left(f_1^+\right)^{k_1}\cdots\left(f_n^+\right)^{k_n}\left(\left[f_1^+,f_2^+\right]\right)^{k_{12}}\left(\left[f_1^+,f_3^+\right]\right)^{k_{13}}\cdots\left(\left[f_{n-1}^+,f_n^+\right]\right)^{k_{n-1,n}}|0\rangle,\\ &k_1,\ldots,k_n,k_{12},k_{13}\ldots,k_{n-1,n}\in\mathbb{Z}_+.\end{aligned} \tag{4.6}$$

Note that $\overline{W}(p)$ is infinite-dimensional. It is in general not an irreducible representation of $\mathfrak{so}(2n+1)$. Let $M(p)$ be the maximal nontrivial submodule of $\overline{W}(p)$. Then the simple module (irreducible representation), corresponding to the parafermion Fock space, is

$$W(p) = \overline{W}(p)/M(p). \tag{4.7}$$

For $\overline{W}(p)$, it is easy to compute its character. Using the fact that the weight of $|0\rangle$ is $\left(-\frac{p}{2},\ldots,-\frac{p}{2}\right)$, the basis (4.6) and the Cauchy identity [18]

$$\frac{1}{\prod_{i=1}^n(1-x_i)\prod_{1\leqslant j<k\leqslant n}(1-x_jx_k)} = \sum_\lambda s_\lambda(x_1,\ldots,x_n) = \sum_\lambda s_\lambda(x), \tag{4.8}$$

where the sum is over all partitions λ and $s_\lambda(x)$ is the Schur symmetric function, one finds

$$\mathrm{char}\,\overline{W}(p) = \frac{(x_1\cdots x_n)^{-p/2}}{\prod_{i=1}^n(1-x_i)\prod_{1\leqslant j<k\leqslant n}(1-x_jx_k)} = (x_1\cdots x_n)^{-p/2}\sum_\lambda s_\lambda(x). \tag{4.9}$$

These Schur functions $s_\lambda(x)$ are the characters of finite-dimensional $\mathfrak{u}(n)$ representations. Hence this expansion yields the branching to $\mathfrak{u}(n)$ of the $\mathfrak{so}(2n+1)$ representation $\overline{W}(p)$. This gives an elegant possibility to label the basis vectors of $\overline{W}(p)$. For each irreducible representation of $\mathfrak{u}(n)$ one can use the corresponding Gelfand–Zetlin basis [1, 8]. The union of all these GZ bases is then the basis for $\overline{W}(p)$. Thus the new basis of $\overline{W}(p)$ consists of vectors of the form (the label p is dropped from the notation of the vectors)

$$|m) \equiv |m)^n \equiv \begin{vmatrix} m_{1n} & \cdots & \cdots & m_{n-1,n} & m_{nn}\\ m_{1,n-1} & \cdots & \cdots & m_{n-1,n-1} & \\ \vdots & \ddots & & & \\ m_{11} & & & & \end{vmatrix} = \begin{pmatrix} [m]^n \\ |m)^{n-1}\end{pmatrix}. \tag{4.10}$$

The top line of this pattern, also denoted by the n-tuple $[m]^n$, is any partition λ (consisting of non-increasing nonnegative numbers). The remaining $n-1$ lines of the pattern will sometimes be denoted by $|m)^{n-1}$. All m_{ij} in the above GZ-pattern are nonnegative integers, satisfying the *betweenness conditions*

$$m_{i,j+1} \geqslant m_{ij} \geqslant m_{i+1,j+1} \qquad (1\leqslant i\leqslant j\leqslant n-1). \tag{4.11}$$

Since the weight of $|0\rangle$ is $\left(-\frac{p}{2},\ldots,-\frac{p}{2}\right)$, the weight of the above vector is determined by

$$h_k|m) = \left(-\frac{p}{2} + \sum_{j=1}^k m_{jk} - \sum_{j=1}^{k-1} m_{j,k-1}\right)|m). \tag{4.12}$$

Now we use the same technique as in [17]. The triple relations (2.4) yield

$$\left[\left[f_i^+,f_j^-\right],f_k^+\right] = 2\delta_{jk}f_i^+.$$

With the identification $\left[f_i^+, f_j^-\right] = 2E_{ij}$ in the standard $\mathfrak{u}(n)$ basis, this is equivalent to the action $E_{ij} \cdot e_k = \delta_{jk} e_i$. Hence the set $\left(f_1^+, f_2^+, \ldots, f_n^+\right)$ is a standard $\mathfrak{u}(n)$ tensor of rank $(1, 0, \ldots, 0)$. So one can attach a unique GZ-pattern with top line $10\cdots0$ to every f_j^+, corresponding to the weight $+\epsilon_j$. Explicitly,

$$f_j^+ \sim \begin{matrix} 10\cdots000 \\ 10\cdots00 \\ \cdots \\ 0\cdots0 \\ \cdots \\ 0, \end{matrix} \tag{4.13}$$

where the pattern consists of $j-1$ zero rows at the bottom, and the first $n-j+1$ rows are of the form $10\cdots0$. The tensor product rule in $\mathfrak{u}(n)$ reads

$$([m]^n) \otimes (10\cdots0) = \left([m]^n_{+1}\right) \oplus \left([m]^n_{+2}\right) \oplus \cdots \oplus \left([m]^n_{+n}\right), \tag{4.14}$$

where $([m]^n) = (m_{1n}, m_{2n}, \ldots, m_{nn})$ and a subscript $\pm k$ indicates an increase of the kth label by ±1:

$$\left([m]^n_{\pm k}\right) = (m_{1n}, \ldots, m_{kn} \pm 1, \ldots, m_{nn}). \tag{4.15}$$

On the right-hand side of (4.14), only those components which are still partitions (i.e., consisting of nondecreasing integers) survive.

Now a general matrix element of f_j^+ can be written as follows [17]:

$$\begin{aligned}(m'|f_j^+|m) &= \left(\begin{matrix}[m]^n_{+k} \\ |m')^{n-1}\end{matrix}\middle| f_j^+ \middle|\begin{matrix}[m]^n \\ |m)^{n-1}\end{matrix}\right) \\ &= \left(\begin{matrix}[m]^n \\ |m)^{n-1};\end{matrix}\middle|\begin{matrix}10\cdots00 \\ 10\cdots0 \\ \cdots \\ 0\end{matrix}\middle|\begin{matrix}[m]^n_{+k} \\ |m')^{n-1}\end{matrix}\right) \times \left([m]^n_{+k} \| f^+ \| [m]^n\right). \end{aligned} \tag{4.16}$$

The first factor on the right-hand side is a $\mathfrak{u}(n)$ Clebsch–Gordan coefficient [17, 28]; the second factor is a reduced matrix element. By the tensor product rule, the first line of $|m')$ has to be of the form (4.15), i.e. $[m']^n = [m]^n_{+k}$ for some k-value.

The special $\mathfrak{u}(n)$ Clebsch–Gordan coefficients (CGCs) appearing here are well known. They can be found, e.g. in [28]. They are expressed by means of $\mathfrak{u}(n)$-$\mathfrak{u}(n-1)$ *isoscalar factors* and $\mathfrak{u}(n-1)$ CGCs, which on their turn are written by means of $\mathfrak{u}(n-1)$-$\mathfrak{u}(n-2)$ isoscalar factors and $\mathfrak{u}(n-2)$ CGCs, etc. The explicit form of the special $\mathfrak{u}(n)$ CGCs appearing here is given in appendix A of [17].

Just as in [17], the main problem is now to find expressions for the reduced matrix elements, i.e. for the functions $G_k([m]^n)$, where

$$G_k([m]^n) = G_k(m_{1n}, m_{2n}, \ldots, m_{nn}) = \left([m]^n_{+k} \| f^+ \| [m]^n\right), \tag{4.17}$$

for arbitrary n-tuples of non-increasing nonnegative integers $[m]^n = (m_{1n}, m_{2n}, \ldots, m_{nn})$. In that case, one can write the explicit actions:

$$f_j^+|m) = \sum_{k,m'} \left(\begin{matrix}[m]^n \\ |m)^{n-1};\end{matrix}\middle|\begin{matrix}10\cdots00 \\ 10\cdots0 \\ \cdots \\ 0\end{matrix}\middle|\begin{matrix}[m]^n_{+k} \\ |m')^{n-1}\end{matrix}\right) G_k([m]^n) \left|\begin{matrix}[m]^n_{+k} \\ |m')^{n-1}\end{matrix}\right), \tag{4.18}$$

$$f_j^-|m) = \sum_{k,m'} \begin{pmatrix} [m]^n_{-k} \\ |m')^{n-1} \end{pmatrix}; \begin{matrix} 10\cdots00 \\ 10\cdots0 \\ \cdots \\ 0 \end{matrix} \begin{vmatrix} [m]^n \\ |m)^{n-1} \end{vmatrix} G_k([m]^n_{-k}) \begin{vmatrix} [m]^n_{-k} \\ |m')^{n-1} \end{vmatrix}. \tag{4.19}$$

The key calculation to determine the unknown functions G_k is to start from the following action:

$$\left[f_n^-, f_n^+\right]|m) = -2h_n|m) = \left(p - 2\left(\sum_{j=1}^{n} m_{jn} - \sum_{j=1}^{n-1} m_{j,n-1}\right)\right)|m). \tag{4.20}$$

Now one can express the left-hand side by means of (4.18) and (4.19), using the explicit form of the CGCs and isoscalar factors (which are rather simple in the case $j = n$). The result is a complicated system of coupled recurrence relations for the functions G_k. It is quite surprising that this system of coupled nonlinear recursion equations has a rather simple solution. Using the relevant boundary conditions, we have been able to solve this system of recurrence relations. This task would have been hardly impossible without the use of Maple. Our main computational result is:

Proposition 4. *The reduced matrix elements G_k appearing in the actions of $f_j^{\pm}$ on vectors $|m)$ of $\overline{W}(p)$ are given by*

$$G_k(m_{1n}, m_{2n}, \ldots, m_{nn}) = \left(-\frac{(\mathcal{E}_n(m_{kn}+n-k)+1)\prod_{j\neq k=1}^{n}(m_{kn}-m_{jn}-k+j)}{\prod_{j\neq \frac{k}{2}=1}^{\lfloor n/2\rfloor}(m_{kn}-m_{2j,n}-k+2j)(m_{kn}-m_{2j,n}-k+2j+1)}\right)^{1/2} \tag{4.21}$$

for k even and by

$$G_k(m_{1n}, m_{2n}, \ldots, m_{nn}) = \left(\frac{(p-m_{kn}+k-1)(\mathcal{O}_n(m_{kn}+n-k)+1)\prod_{j\neq k=1}^{n}(m_{kn}-m_{jn}-k+j)}{\prod_{j\neq \frac{k+1}{2}=1}^{\lceil n/2\rceil}(m_{kn}-m_{2j-1,n}-k+2j-1)(m_{kn}-m_{2j-1,n}-k+2j)}\right)^{1/2} \tag{4.22}$$

for k odd.

Herein $\mathcal{E}$ and $\mathcal{O}$ are the even and odd functions defined by

$$\begin{aligned} \mathcal{E}_j &= 1 \quad \text{if} \quad j \text{ is even and 0 otherwise,} \\ \mathcal{O}_j &= 1 \quad \text{if} \quad j \text{ is odd and 0 otherwise,} \end{aligned} \tag{4.23}$$

where obviously $\mathcal{O}_j = 1 - \mathcal{E}_j$, but it is still convenient to use both notations. Also, note that products such as $\prod_{j\neq k=1}^{n}$ mean 'the product over all j-values running from 1 to n, but excluding $j = k$'. The notation $\lfloor a\rfloor$ (resp. $\lceil a\rceil$) refers to the *floor* (resp. *ceiling*) of a, i.e. the largest integer not exceeding a (resp. the smallest integer greater than or equal to a).

To present all the details of this computational result is unrealistic. Just as in [17], the proof consists of verifying that all triple relations (2.4) hold when acting on any vector $|m)$. Each such verification leads to an algebraic identity in the n variables of the partition, $m_{1n}, \ldots, m_{nn}$. In such computations, there are some intermediate verifications: e.g. the action $\left[f_j^+, f_k^-\right]|m)$ should leave the top row of the GZ-pattern $|m)$ invariant (since $\left[f_j^+, f_k^-\right]$ belongs to $\mathfrak{u}(n)$). In fact, it must give (up to a factor 2) the known action of the standard $\mathfrak{u}(n)$ matrix elements E_{jk} in the classical GZ-basis.

The explicit expressions for the reduced matrix elements give the action of the generators in the basis of $\overline{W}(p)$, for arbitrary p. The structure of the maximal submodule $M(p)$ and hence of the irreducible factor module $W(p)$ is revealed by examining when these matrix elements vanish. It follows from (4.21) and (4.22) that the only crucial factor is

$$(p - m_{kn} + k - 1), \qquad (k \text{ odd}).$$

In particular for $k = 1$ this factor is $(p - m_{1n})$, and m_{1n} is the largest integer in the GZ-pattern. Starting from the vacuum vector, with a GZ-pattern consisting of all zeros, one can raise the entries in the GZ-pattern by applying the operators f_j^+. However, when m_{1n} has reached the value p it can no longer be increased. As a consequence, all vectors $|m)$ with $m_{1n} > p$ belong to the submodule $M(p)$. This uncovers the structure of $W(p)$:

Corollary 5. *For p a positive integer, the parafermion Fock space $W(p)$ has as a basis the vectors $|m)$ with $m_{1n} \leqslant p$. In other words, the top line of $|m)$ is a partition λ with largest part not exceeding p, i.e. $\ell(\lambda') \leqslant p$. As a consequence, it follows from (4.9) that*

$$\operatorname{char} W(p) = (x_1 \cdots x_n)^{-p/2} \sum_{\lambda, \ell(\lambda') \leqslant p} s_\lambda(x_1, \ldots, x_n).$$

This last result coincides with (3.9). Note that p must be a positive integer in order to have a positive inner product $(m|m)$ for all possible patterns.

Of course, we did not make these hard computations just to find the character of $W(p)$, which was already determined by standard techniques. Our main purpose was to find a proper orthonormal basis for $W(p)$, in which the action of the parafermion operators $f_j^\pm$ can be computed explicitly. This has now been obtained by means of (4.18), (4.19) and (4.21), (4.22). Let us summarize this result, inserting also the explicit CGCs of [17, appendix A].

Theorem 6. *An orthonormal basis for the parafermion Fock space $W(p)$ is given by the vectors $|m)$, see (4.10), with $m_{1n} \leqslant p$. Its dimension is given by (3.11). The action of the Cartan algebra elements of $\mathfrak{so}(2n+1)$ is*

$$h_k|m) = \left(-\frac{p}{2} + \sum_{j=1}^{k} m_{jk} - \sum_{j=1}^{k-1} m_{j,k-1} \right) |m), \qquad (1 \leqslant k \leqslant n). \tag{4.24}$$

The action of the parafermion operators $f_j^\pm$ is given below, first for $j = n$ (simple case) and then for general j. We have

$$f_n^+|m) = \sum_{i=1}^{n} \left(\frac{\prod_{k=1}^{n-1}(l_{k,n-1} - l_{in} - 1)}{\prod_{k\neq i=1}^{n}(l_{kn} - l_{in})} \right)^{1/2} G_i(m_{1n}, m_{2n}, \ldots, m_{nn})|m)_{+in} \tag{4.25}$$

$$f_n^-|m) = \sum_{i=1}^{n} \left(\frac{\prod_{k=1}^{n-1}(l_{k,n-1} - l_{in})}{\prod_{k\neq i=1}^{n}(l_{kn} - l_{in} + 1)} \right)^{1/2} G_i(m_{1n}, \ldots, m_{in} - 1, \ldots, m_{nn})|m)_{-in}. \tag{4.26}$$

Herein, $l_{ij} = m_{ij} - i$, G_i is determined by (4.21), (4.22), and $\pm i$, n attached as a subscript to $|m)$ indicates a replacement $m_{i,n} \to m_{i,n} \pm 1$. In general,

$$\begin{aligned} f_j^+|m) = \sum_{i_n=1}^{n} \sum_{i_{n-1}=1}^{n-1} \cdots \sum_{i_j=1}^{j} S(i_n, i_{n-1}) S(i_{n-1}, i_{n-2}) \cdots S(i_{j+1}, i_j) \left(\frac{\prod_{k=1}^{j-1}(l_{k,j-1} - l_{i_j,j} - 1)}{\prod_{k\neq i_j=1}^{j}(l_{kj} - l_{i_j,j})} \right)^{1/2} \\ \times \prod_{r=1}^{n-j} \left(\frac{\prod_{k\neq i_{n-r}=1}^{n-r}(l_{k,n-r} - l_{i_{n-r+1},n-r+1} - 1) \prod_{k\neq i_{n-r+1}=1}^{n-r+1}(l_{k,n-r+1} - l_{i_{n-r},n-r})}{\prod_{k\neq i_{n-r+1}=1}^{n-r+1}(l_{k,n-r+1} - l_{i_{n-r+1},n-r+1}) \prod_{k\neq i_{n-r}=1}^{n-r}(l_{k,n-r} - l_{i_{n-r},n-r} - 1)} \right)^{1/2} \\ \times G_{i_n}(m_{1n}, m_{2n}, \ldots, m_{nn})|m)_{+i_n,n;+i_{n-1},n-1;\ldots;+i_j,j}; \end{aligned} \tag{4.27}$$

$$f_j^-|m) = \sum_{i_n=1}^{n}\sum_{i_{n-1}=1}^{n-1}\cdots\sum_{i_j=1}^{j} S(i_n,i_{n-1})S(i_{n-1},i_{n-2})\cdots S(i_{j+1},i_j)\left(\frac{\prod_{k=1}^{j-1}(l_{k,j-1}-l_{i_j,j})}{\prod_{k\neq i_j=1}^{j}(l_{kj}-l_{i_j,j}+1)}\right)^{1/2}$$
$$\times\prod_{r=1}^{n-j}\left(\frac{\prod_{k\neq i_{n-r}=1}^{n-r}(l_{k,n-r}-l_{i_{n-r+1},n-r+1})\prod_{k\neq i_{n-r+1}=1}^{n-r+1}(l_{k,n-r+1}-l_{i_{n-r},n-r}+1)}{\prod_{k\neq i_{n-r+1}=1}^{n-r+1}(l_{k,n-r+1}-l_{i_{n-r+1},n-r+1}+1)\prod_{k\neq i_{n-r}=1}^{n-r}(l_{k,n-r}-l_{i_{n-r},n-r})}\right)^{1/2}$$
$$\times G_{i_n}(m_{1n},\ldots,m_{i_n,n}-1,\ldots,m_{nn})|m)_{-i_n,n;-i_{n-1},n-1;\ldots;-i_j,j}. \tag{4.28}$$

Once again, each symbol $\pm i_k, k$ *attached as a subscript to* $|m)$ *indicates a replacement* $m_{i_k,k} \to m_{i_k,k} \pm 1$, *and*

$$S(k,l) = \begin{cases} 1 & \text{for} \quad k \leqslant l \\ -1 & \text{for} \quad k > l. \end{cases} \tag{4.29}$$

5. Example: $W(p)$ for $\mathfrak{so}(5)$

The above formulae for general n look rather involved, so perhaps it is useful to give an example. Let us consider the case $n = 2$, i.e. the Lie algebra $\mathfrak{so}(5)$. For any positive integer p, the parafermion Fock space $W(p)$ has dimension given by (3.11),

$$\dim W(p) = \frac{(p+3)(p+2)(p+1)}{6}. \tag{5.1}$$

The set of orthonormal basis vectors of $W(p)$ is given by all vectors with integer m-patterns of the form

$$\begin{vmatrix} m_{12}, m_{22} \\ m_{11} \end{vmatrix}, \qquad p \geqslant m_{12} \geqslant m_{11} \geqslant m_{22} \geqslant 0. \tag{5.2}$$

Clearly, the number of different m-patterns satisfying these inequalities coincides with (5.1). The action of the parafermion generators f_1^+ and f_2^+ is given by

$$f_1^+\begin{vmatrix} m_{12}, m_{22} \\ m_{11} \end{vmatrix} = \sqrt{(m_{11}-m_{22}+1)(p-m_{12})}\begin{vmatrix} m_{12}+1, m_{22} \\ m_{11}+1 \end{vmatrix}$$
$$-\sqrt{(m_{12}-m_{11})(m_{22}+1)}\begin{vmatrix} m_{12}, m_{22}+1 \\ m_{11}+1 \end{vmatrix}, \tag{5.3}$$

$$f_2^+\begin{vmatrix} m_{12}, m_{22} \\ m_{11} \end{vmatrix} = \sqrt{(m_{12}-m_{11}+1)(p-m_{12})}\begin{vmatrix} m_{12}+1, m_{22} \\ m_{11} \end{vmatrix}$$
$$+\sqrt{(m_{11}-m_{22})(m_{22}+1)}\begin{vmatrix} m_{12}, m_{22}+1 \\ m_{11} \end{vmatrix}. \tag{5.4}$$

The action of f_1^- and f_2^- follows from the above expressions using $(m'|f_j^-|m) = (m|f_j^+|m')$. In this case, it is not too difficult to check these actions by hand: one can do this by comparing the action of the left-hand side of the triple relation (2.4) with the action of the right-hand side of (2.4) (for some values of j, k, l and ξ, η, ϵ).

6. Summary and conclusion

In this paper we have given a complete description of the unitary irreducible representations $W(p)$ of $\mathfrak{so}(2n+1)$ with highest weight $\left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}\right)$, which in particular are of interest in the theory of parafermion statistics because these representations are exactly the Fock spaces

of n parafermions, with p being the order of the parastatistics. Although many aspects of $W(p)$ are known from general Lie algebra representation theory, the problem of giving an appropriate orthogonal basis for $W(p)$ and the explicit action of the parafermion operators on such a basis was not solved. In the present paper, we solve this problem using some group theoretical methods and computational techniques. A crucial role in our analysis is played by the $\mathfrak{u}(n)$ subalgebra of $\mathfrak{so}(2n+1)$, generated by all commutators of the parafermion creation and annihilation operators. Taking a certain parabolic subalgebra $\mathcal{P}$ containing $\mathfrak{u}(n)$ and a trivial module of $\mathcal{P}$ generated from the vacuum, i.e. the lowest weight vector of weight $\left(-\frac{p}{2}, -\frac{p}{2}, \ldots, -\frac{p}{2}\right)$, an induced module $\overline{W}(p)$ of $\mathfrak{so}(2n+1)$ is constructed. The Fock module $W(p)$ is the quotient of this induced module by its maximal submodule $M(p)$. The character of the induced module is readily obtained and by a classical result of Cauchy–Littlewood the characters can be rewritten as an infinite sum over partitions of Schur symmetric functions. This can be reinterpreted as a decomposition of the $\mathfrak{so}(2n+1)$ module into an infinite sum of finite-dimensional simple $\mathfrak{u}(n)$ modules labeled by partitions. For each irreducible representation of $\mathfrak{u}(n)$ one can use the corresponding Gelfand–Zetlin basis. The union of all these GZ-basis vectors is the basis for the induced module $\overline{W}(p)$. The main calculation is then the action of the parafermion operators on this basis. Just as for the case of paraboson operators [17], the collection of the n parafermion creation operators is a $\mathfrak{u}(n)$ tensor operator of rank $(1, 0, \ldots, 0)$. To calculate its matrix elements, they are written as a product of certain $\mathfrak{u}(n)$ Clebsch–Gordan coefficient and a reduced matrix element. As the relevant $\mathfrak{u}(n)$ CGCs are known, the problem is to find the reduced matrix elements. Solving a set of recurrence relations for these leads to their expressions. These explicit expressions give not only the action of the generators in the basis of $\overline{W}(p)$, they also yield the structure of the maximal submodule $M(p)$ and hence of the irreducible factor module $W(p)$. This leads to the main result of the paper: an explicit basis of $W(p)$ (consisting of all possible GZ-patterns with integer entries at most p) and the explicit action of the generators in this basis. As an illustration, the case $n = 2$ is given in more detail.

We have considered here the parafermion Fock spaces of order p for a finite degree of freedom (n finite). The real interest lies in such quantum systems (parabosons and parafermions) with an infinite degree of freedom ($n = +\infty$), see e.g. [11, 16]. In a forthcoming paper, we hope to report on how the results of [17] and of the current paper can be used to construct representations of order p for an infinite set of parabosons and parafermions. Also an investigation of representations of the 'parastatistics algebra' in which both parabosons and parafermions appear [19] should be of interest.

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